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Newtonian Gravitational Field Theory: Two-Body Problem

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Abstract

The effects of a dual force which appears in a consistent field theory of Newtonian gravitation are explored by a study of the motion of two bodies which interact with each other through the gravitational field. The equations of motion are solved exactly. Among the results obtained, we find that the present theory formulated in accordance with the Special Theory of Relativity leads to the same analytical result for the precession of the perihelion of the orbit as does Einstein's General Theory of Relativity. Another result is that classical particles are endowed with an intrinsic angular momentum of constant magnitude—a helicity of classical origin. Other results, such as the period of revolution, are similar to Kepler's law, except for relativistic corrections. A slight deviation from the planar orbit of classical theory results, and may be observable. This deviation is related to the magnitude of the precession of the perihelion of the orbit. The significance of these results for charged particles, viewed classically or quantum mechanically, are discussed.

1. *Introduction*

A previous publication (Schwebel, 1970a), developed a description for Newtonian gravitation similar to Maxwell's field theory of electromagnetism. The gravitational field was found to be longitudinal and to be propagated with the velocity of light. At the same time, a consistent formulation of a field theory for each phenomenon gave rise to a mathematical formalism which eliminated the difficulties usually associated with field theories, such as self-force, self-energy, etc.

A feature of the consistent field theory which is absent from conventional theory is the appearance of an additional interaction term, which has been named a dual force. It is the purpose of the present paper to explore the consequences due to the existence of the dual force.

We will apply the theory to the gravitational interaction between two particles. It will be found that the dual force will lead to the same analytical result for the precession of the perihelion of the orbit of one particle about the other as is obtained from Einstein's General Theory of Relativity. Another result is that classical particles have an intrinsic spin or helicity which is orthogonal to its orbital angular momentum. Moreoever, the vector sum of both angular momenta is a constant. We will also find that under the influence of the dual force the orbit of the gravitating particle will lie on the surface of a cone whose vertex is occupied by the second particle. The magnitude of the semi-angle of the cone is directly related to the value for the precession of the perihelion of the orbit.

Our first concern will be the equations of motion and their exact solution. This will be followed by calculations of the precession of the perihelion of the orbit and of its period of revolution. Finally, we discuss the results obtained and their significance.

2. *Equations of Motion*

We consider the gravitational interaction between two point particles. Since the universe of discourse is limited to just these particles, we take one of the particles as the origin of the coordinate system and consider the second particle orbiting about the first (Schwebel, 1970b). Therefore, the second particle is in motion in the static field of the first particle.

It follows, using the notation and results of Schwebel (1970a), that the equations of motion are

$$\dot{\mathbf{p}}_2 = \mu_2 \mathbf{N}_1 + \lambda\mu_2(\mathbf{v}_2 \times \mathbf{N}_1); \qquad (c = 1) \tag{2.1a}$$

$$\dot{p}_2{}^0 = \mu_2 \mathbf{v}_2 \cdot \mathbf{N}_1 \tag{2.1b}$$

where

$\mu = \sqrt{(-G)}m$ ($G \equiv$ gravitational constant);
$\mathbf{p}_2$ = momentum vector of particle two;
$p_2{}^0$ = time component of the four-momentum vector;
$\mathbf{N}_1$ = gravitational field due to particle one;
$= \mu_1 \mathbf{r}/r^3$;
$\mu_2 \mathbf{v}_2 \times \mathbf{N}_1$ = dual Newtonian force

These equations include the dual Newtonian force which is weighted with a multiplicative constant, λ.

The field exerted by particle one on particle two is given by the expression

$$\mathbf{N}_1 = \mu_1 \mathbf{r}/r^3 \tag{2.2}$$

where $\mathbf{r}$ is the relative displacement vector of particle two from particle one. Inserting this value for $\mathbf{N}_1$ into equation (2.1a) and taking the cross-product of the latter with $\mathbf{r}$, we obtain the result.

$$\mathbf{r} \times \mathbf{p}_2 - \lambda\mu_1 \mu_2(\mathbf{r}/r) = \mathbf{L} = \text{const.} \tag{2.3}$$

The conservation of angular momentum which this relation expresses differs from the conventional law. There is an additional contribution to the orbital angular momentum term of an intrinsic angular momentum which arises from the dual Newtonian force. The intrinsic angular momentum is constant in magnitude and is directed along the displacement vector $\mathbf{r}$. Note that if the momentum vector $\mathbf{p}_2$ is parallel to $\mathbf{r}$, then the total angular momentum of the system is the intrinsic angular momentum and we have the system displaying a classical type of helicity.

If equation (2.2) is inserted into equation (2.1b), we find the relation which is recognized as the conservation of energy of the system:

$$p_2{}^0 + \frac{\mu_1 \mu_2}{r} = E = \text{const.} \tag{2.4}$$

We see that the dual force does not explicitly contribute to the total energy.

3. *Determination of the Orbit*

If we introduce a spherical coordinate system with the polar axis in the direction of **L**, then equation (2.3) becomes

$$\gamma m_2[-r^2 \dot{\theta} \sin\phi - r^2 \sin\theta \cos\theta \cos\phi\dot{\phi}] - \lambda\mu_1 \mu_2 \sin\theta \cos\phi = 0$$

$$\gamma m_2[r^2 \cos\phi\dot{\theta} - r^2 \sin\theta \cos\theta \sin\phi\dot{\phi}] - \lambda\mu_1 \mu_2 \sin\theta \sin\phi = 0$$

and

$$\gamma m_2[r^2 \dot{\phi} \sin^2\theta] - \lambda\mu_1 \mu_2 \cos\theta = L$$

where $L = |\mathbf{L}|$ and $\gamma = \{1 - v_2{}^2\}^{-1/2}$. We have introduced the relativistic values for $\mathbf{p}_2$ into equation (2.3) to obtain the above equations.

Some elementary algebraic calculations enable us to obtain from these equations the following two relations.

$$\cos\theta = -\lambda\mu_1 \mu_2/L \tag{3.1}$$

and

$$\gamma m_2 r^2 \dot{\phi} = L \tag{3.2}$$

From equation (2.4), we obtain, using the relativistic value for $p_2{}^0$, the relation

$$\gamma m_2 + \frac{\mu_1 \mu_2}{r} = E = \text{const.} \tag{3.3}$$

Equation (3.1) states that θ is a constant. If λ were equal to zero, then $\theta = \pi/2$ and the orbit would lie in a plane perpendicular to the angular momentum vector **L**—the classical solution. We find that λ does not equal zero, though the right-hand side of equation (3.1) will be very small. For that reason, the orbit lies on the surface of a cone. The cone has particle one at its vertex, a semi-angle of θ and an axis parallel to the vector **L**.

Using the relation

$$v_2{}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2 = 1 - \frac{1}{\gamma^2}$$

in conjunction with equations (3.1), (3.2) and (3.3), we find

$$\dot{r} = \pm\{(Er - \mu_1 \mu_2)^2 - m_2{}^2 r^2 - L^2 \sin^2\theta\}^{1/2}/|Er - \mu_1 \mu_2| \tag{3.4}$$

and

$$\dot{\phi} = L/(\gamma m_2 r^2) \tag{3.5}$$

The last two equations yield the result

$$\pm\frac{d\phi}{L}=\frac{dr}{r\{(Er-\mu_1\mu_2)^2-m_2{}^2r^2-L^2\sin^2\theta\}^{1/2}}$$

Integration yields

$$\frac{1}{r}=A[1+B\sin\beta(\phi-\phi_0)] \tag{3.6}$$

where the choice of sign is unimportant (since it merely denotes a choice of phase), and

$$A=(\mu_1\mu_2E)/(\mu_1{}^2\mu_2{}^2-L^2\sin^2\theta)=-E\cos\theta/L\lambda\left(\frac{\cos^2\theta}{\lambda^2}-\sin^2\theta\right)$$

$$B=\{m_2{}^2+\lambda^2\tan^2\theta(E^2-m_2{}^2)\}^{1/2}/E$$

$$\beta=\{\sin^2\theta-(\cos^2\theta/\lambda^2)\}^{1/2}$$

4. *The Period and the Precession of the Perihelion of the Orbit*

The advance of the perihelion of the orbit of the gravitating particle is obtained from the argument of the trigonometric function in equation (3.6). For, successive minima of the distance between the two particles occur when the change in phase is 2π, i.e., when

$$\Delta\beta(\phi-\phi_0)=\beta\Delta\phi=2\pi$$

Therefore the advance of the perihelion, δ, is given by

$$\delta=\frac{2\pi}{\beta}-2\pi=2\pi\left(\frac{1}{\beta}-1\right)$$

Since $\beta<1$, it follows that the perihelion advances as the particle pursues its orbit. Using equation (3.1), the value for β and the information that $\cos\theta$ is very small, we find to a good approximation that

$$\delta=\pi\left(1+\frac{1}{\lambda^2}\right)\lambda^2\mu_1{}^2\mu_2{}^2/L^2 \tag{4.1}$$

If $\lambda^2=5$, then δ is precisely the result obtained from Einstein's General Theory of Relativity.

The calculation of the period of the orbit proceeds from equations (3.2) and (3.6). The result is that

$$T=\left[\frac{E}{1-B^2}-\mu_1\mu_2A\right]\frac{2\pi}{L\beta A^2(1-B^2)^{1/2}}$$

which simplifies if we introduce the 'semi-major' axis, a, defined by

$$a\equiv\tfrac{1}{2}(r(\max.)+r(\min.))$$

Using equation (3.6) and the definitions for A, B, and β, we can express the period, T, as a function of the 'semi-major' axis and obtain

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_1}} \left(\frac{m_2}{E}\right)^{3/2} \tag{4.2}$$

The 'semi-major' axis, a, can be expressed in physical terms

$$a = \frac{Gm_1 m_2 E}{m_2{}^2 - E^2} \tag{4.3}$$

Another form for equation (4.1) is

$$\delta = \frac{24\pi^3 a^2}{c^2 T^2(1 - B^2)} \left(\frac{m_2 c^2}{E}\right)^4; \qquad (\lambda^2 = 5) \tag{4.4}$$

Equations (4.1), (4.2), (4.3) and (4.4) are for bound orbits for which $E^2 - m_2{}^2 c^4 < 0$. We have reintroduced c into the last equation to make comparison easier with results found elsewhere.

5. *Discussion*

The inclusion of the dual force in the equations of motion for two particles has led to a number of results. First, as equation (2.3) displays, there is an intrinsic angular momentum about the direction of the displacement of one particle from the other—a sort of classical helicity. Secondly, we find that the advance of the perihelion of the orbit of one particle about the other is given by a relation, equation (4.1), which differs from that obtained from Einstein's General Theory of Relativity only by a multiplicative constant. The value of this constant depends solely on the constant λ which was introduced as a measure of the role of the dual force relative to the conventional Newtonian gravitational force. Thirdly, the trajectory of one particle in motion about a second particle differs qualitatively though practically negligibly from the classical Kepler orbits.

Because λ is non-zero, we find that the trajectory of the particle in motion lies on the surface of a cone with the second particle at its apex. The magnitude of the semi-angle of the cone is so large—approximately 90°—that the cone is practically indistinguishable from a plane. Nevertheless, it may be possible to observe the departure of the orbit of the particle, say the earth, from planar orbit predicted by conventional theory. According to the above results, the earth's orbit should lie on a cone with the sun at its apex. If the background of the stars is photographed from the earth in the direction of the sun–earth axis, and if two such photographs taken at azimuths 180° apart are compared, then there should be a shift in the positions of the stars. The angular shift should be equal to twice the complement of the semi-angle of the cone, an exceedingly small angle to measure. Another consequence to be expected is that those planets for which the advance of their perihelions are large should have orbits that lie farther from the planar orbit predicted by classical Newtonian theory.

Both the above effects are small and the observation problem is complicated by the interactions of the other planets with the sun–earth system. The above calculations have not included such perturbations.

In classical theory, equation (4.2) is amended by the introduction of the reduced mass of the system. The procedure effectively replaces m_1 with $m_1 + m_2$. Of course, the factor $(m_2/E)^{3/2}$ is relativistic in origin and does not appear in the classical result. To obtain the corresponding correction for the relativistic result obtained, equation (4.2), we must resort to relational mechanics (Schwebel, 1970b) in which the two-particle system is treated as a physical entity. Using the form of Newton's equations derived there, we find that in equation (4.2) we must replace m_1 with $m_1 + m_2$ and in the factor (m_2/E), ${}^{3/2} m_2$ must be replaced by the reduced mass $\sigma \equiv (m_1 m_2)/(m_1 + m_2)$. Similar changes occur throughout; for example, equation (4.3) undergoes the change

$$a = (Gm_1 m_2 E)/(\sigma^2 - E^2)$$

and equation (4.4) becomes

$$\delta = \frac{24\pi^3 a^2}{c^2 T^2(1 - B^2)}\left(\frac{\sigma c^2}{E}\right)^4$$

The enumerated consequences given above of the effects of the dual force are corroborative of the reality of such a force as well as indicative of the value of the gravitational field theory of which it forms an integral part. That field theory was developed along with electromagnetic field theory and we can anticipate similar results for the interaction of two electrically charged particles. Thus, indeed, it is a straightforward procedure to show that an intrinsic angular momentum proportional to e^2/c will appear as well as the other effects which were found for two mass particles. In particular, for charged particles, the quantum mechanical transcript of equation (2.3) with its promise of accounting for the anomalous magnetic moment of the particles is an intriguing application which will be the subject of another paper.

References

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